

THE (b, c) -INVERSE IN RINGS AND IN THE BANACH CONTEXT

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ABSTRACT. In this article the (b, c) -inverse will be studied. Several equivalent conditions for the existence of the (b, c) -inverse in rings will be given. In particular, the conditions ensuring the existence of the (b, c) -inverse, of the annihilator (b, c) -inverse and of the hybrid (b, c) -inverse will be proved to be equivalent, provided b and c are regular elements in a unitary ring \mathcal{R} . In addition, the set of all (b, c) -invertible elements will be characterized and the reverse order law will be also studied. Moreover, the relationship between the (b, c) -inverse and the Bott-Duffin inverse will be considered. In the context of Banach algebras, integral, series and limit representations will be given. Finally the continuity of the (b, c) -inverse will be characterized.

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1. INTRODUCTION

Given a semigroup \mathcal{S} and elements $a, b, c \in \mathcal{S}$, the notion of a (b, c) -inverse of a was introduced in [5] (see [5, Definition 1.3] or section 2). This inverse is a class of outer inverses which encompasses several well known generalized inverses, such as the Drazin inverse and the Moore-Penrose inverse (in the presence of an involution). Furthermore, the aforementioned inverse has been studied by several authors (see [5, 6, 7, 4, 8]) and it is closely related to other outer inverses such as the $a_{p,q}^{(2,l)}$ outer inverse in Banach algebras and the image-kernel (p, q) -inverse in rings (see [3, 7, 11] and sections 2 and 7).

The aim of this article is to study further properties of the (b, c) -inverse enlarging the underlying set, in particular, considering the contexts of unitary rings and Banach algebras. In section 3, after having recalled some preliminary definitions and facts in section 2, further equivalent conditions that ensure the existence of the outer inverse under consideration will be given. From this result it will be derived that the (b, c) -inverse, the annihilator (b, c) -inverse and the hybrid (b, c) -inverse (see [5]) coincide, when b and c are regular. In section 4 the set of all (b, c) -invertible elements in a unitary ring will be characterized and more equivalent characterizations of the (b, c) -inverse will be proved. In section 5 the relationship between the (b, c) -inverse and the Bott-Duffin (p, q) -inverse (see [5, Definition 3.2]) will be studied. In fact, on the one hand, the latter inverse is a particular case of the former inverse (when b and c are idempotents), but on the other, it will be proved that (b, c) -invertible elements are Bott-Duffin (p, q) -invertible for suitable p and q . In section 6 the reverse order law for the (b, c) -inverse and the Bott-Duffin (p, q) -inverse will be studied. Finally, in

section 7 several results concerning integral representations, convergent series, limits, approximation and continuity of the (b, c) -inverse will be presented.

2. PRELIMINARY DEFINITIONS AND FACTS

From now on \mathcal{R} will denote a unitary ring with unity 1. Let \mathcal{R}^{-1} (respectively \mathcal{R}^\bullet) be the set of invertible elements (respectively of idempotents) of \mathcal{R} . Given $a \in \mathcal{R}$, the *image ideals* are defined by $a\mathcal{R} := \{ax : x \in \mathcal{R}\}$ and $\mathcal{R}a := \{xa : x \in \mathcal{R}\}$, and the *kernel ideals* by $a^{-1}(0) := \{x \in \mathcal{R} : ax = 0\}$ and $a_{-1}(0) := \{x \in \mathcal{R} : xa = 0\}$.

An element $a \in \mathcal{R}$ is said to be *regular*, if there exists $x \in \mathcal{R}$ such that $a = axa$. The element x , which is not uniquely determined by a , will be said to be a *generalized inverse* or an *inner inverse* of a . The set of regular elements of \mathcal{R} will be denoted by $\hat{\mathcal{R}}$ and, given $v \in \hat{\mathcal{R}}$, $v\{1\}$ will stand for the set of all inner inverses of v . In addition, if $y \in \mathcal{R}$ satisfies $yay = y$, then y is said to be an *outer inverse* of a . An element $z \in \mathcal{R}$ is said to be a *normalized generalized inverse* of a , if z is an inner inverse and an outer inverse of a . Note that if $w \in \mathcal{R}$ is an inner inverse of a , then $w' = waw$ is a normalized generalized inverse of a .

Now the definition of the key notion of this article will be recalled.

Definition 2.1 ([5, Definition 1.3]). *Let \mathcal{R} be a ring with unity and consider $b, c \in \mathcal{R}$. The element $a \in \mathcal{R}$ will be said to be (b, c) -invertible, if there exists $y \in \mathcal{R}$ such that the following equations hold:*

- (i) $y \in (b\mathcal{R}y) \cap (y\mathcal{R}c)$,
- (ii) $b = yab, c = cay$.

If such inverse exists, then it is unique (see [5, Theorem 2.1 (i)]). Thus in what follows, if the element y in Definition 2.1 exists, then it will be denoted by $a^{-(b, c)}$. In addition, according to [5, Theorem 2.1 (ii)], $a^{-(b, c)}$ is an outer inverse of a .

In the following remark some properties of b and c will be considered (see also [7]).

Remark 2.2. Let \mathcal{R} be a ring with unity and let $a, b, c \in \mathcal{R}$. Recall that according to [5, Proposition 6.1], necessary and sufficient for a to be (b, c) -invertible is that there exists $y \in \mathcal{R}$ such that $yay = y$, $y\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}y = \mathcal{R}c$.

- (i) Let $b', c' \in \mathcal{R}$ be such that $b'\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}c' = \mathcal{R}c$. Then, from [5, Proposition 6.1], a is (b, c) -invertible if and only if a is (b', c') -invertible. Moreover, in this case $a^{-(b', c')} = a^{-(b, c)}$.

Next suppose that $a^{-(b, c)}$ exists.

- (ii) Let $z \in \mathcal{R}$. Recall that necessary and sufficient for z to have a right inverse (respectively a left inverse) is that $z\mathcal{R} = \mathcal{R}$ (respectively $\mathcal{R}z = \mathcal{R}$). Consequently, applying again [5, Proposition 6.1], necessary and sufficient for $a \in \mathcal{R}^{-1}$ is that b is right invertible and c is left invertible. In this case, $a^{-(b, c)} = a^{-1}$.

- (iii) The elements b and c are regular. In fact, according to Definition 2.1, there are $g, h \in \mathcal{R}$ such that $bga^{-(b, c)} = a^{-(b, c)} = a^{-(b, c)}hc$. Therefore,

$$\begin{aligned} b &= a^{-(b, c)}ab = bga^{-(b, c)}ab = bgb, \\ c &= caa^{-(b, c)} = caa^{-(b, c)}hc = chc. \end{aligned}$$

(iv) Let $g' = gbg$ and $h' = hch$. Then, g' and h' are normalized generalized inverses of g and h , respectively. In addition, since according to Definition 2.1, there is $z \in \mathcal{R}$ such that $a^{-(b, c)} = bzc$, $a^{-(b, c)} = bg'a^{-(b, c)}$ and $a^{-(b, c)} = a^{-(b, c)}h'c$. In fact,

$$\begin{aligned} bg'a^{-(b, c)} &= bg'bzc = bzc = a^{-(b, c)}. \\ a^{-(b, c)}h'c &= bzh'c = bzc = a^{-(b, c)}. \end{aligned}$$

Therefore, in Definition 2.1, not only $b, c \in \hat{\mathcal{R}}$, but also the elements that satisfy condition (i) of Definition 2.1 (g and h in (iii)) can be chosen as normalized generalized inverses of b and c .

A particular case of the (b, c) -inverse is the Bott-Duffin inverse.

Definition 2.3 ([5, Definition 3.2]). *Let \mathcal{R} be a ring with unity and consider $p, q \in \mathcal{R}^\bullet$. The element $a \in \mathcal{R}$ will be said to be Bott-Duffin (p, q) -invertible, if there exists $y \in \mathcal{R}$ such that*

- (i) $y = py = yq$,
- (ii) $yap = p$ and $qay = q$.

Clearly, given $p, q \in \mathcal{R}^\bullet$, the Bott-Duffin (p, q) -inverse is nothing but the (b, c) -inverse when b and c are idempotents. In addition, since there exists at most one (b, c) -inverse, the Bott-Duffin inverse is unique, if it exists. According to what has been said, if $a \in \mathcal{R}$ is Bott-Duffin (p, q) -invertible, then the element y in Definition 2.3 will be denoted by $a^{-(p, q)}$. On the other hand, it is worth noticing that the image-kernel (p, q) -inverse (see [7, 11]) is closely related to the Bott-Duffin (p, q) -inverse. In fact, according to [7, Proposition 3.4], this inverse coincides with the Bott-Duffin $(p, 1 - q)$ -inverse.

In the article [5] two other outer inverses which are related to the (b, c) -inverse were considered. Next their definitions will be recalled.

Definition 2.4 ([5, Definitions 6.2-6.3]). *Let \mathcal{R} be a ring with unity and consider $a, b, c \in \mathcal{R}$.*

- (i). *The element $y \in \mathcal{R}$ will be said to be an annihilator (b, c) -inverse of a , if*

$$y = yay, \quad y_{-1}(0) = b_{-1}(0), \quad y^{-1}(0) = c^{-1}(0).$$

- (ii). *The element $y \in \mathcal{R}$ will be said to be a hybrid (b, c) -inverse of a , if*

$$y = yay, \quad y\mathcal{R} = b\mathcal{R}, \quad y^{-1}(0) = c^{-1}(0).$$

In the same conditions of Definition 2.4, according to [5, Theorem 6.4], there can be at most one annihilator (respectively hybrid) (b, c) -inverse of $a \in \mathcal{R}$. Moreover, it is not difficult to prove that if a has a (b, c) -inverse, then a has a hybrid inverse, which in turn implies that a has an annihilator inverse.

The last generalized inverse in the context of rings that it is necessary to recall for this article is the group inverse. Let \mathcal{R} be a ring with unity and consider $a \in \mathcal{R}$. The element a will be said to be *group invertible*, if there exists $b \in \mathcal{R}$ such that

$$a = aba, \quad b = bab, \quad ab = ba.$$

It is well known that the group inverse is unique, if it exists. In that case, the group inverse of $a \in \mathcal{R}$ will be denoted by a^\sharp . Note that $aa^\sharp, a^\sharp a \in \mathcal{R}^\bullet$ and

$$\begin{aligned} a\mathcal{R} &= aa^\sharp\mathcal{R} = a^\sharp a\mathcal{R} = a^\sharp\mathcal{R}, \\ \mathcal{R}a &= \mathcal{R}a^\sharp a = \mathcal{R}aa^\sharp = \mathcal{R}a^\sharp, \\ a^{-1}(0) &= (a^\sharp a)^{-1}(0) = (aa^\sharp)^{-1}(0) = (a^\sharp)^{-1}(0), \\ a_{-1}(0) &= (aa^\sharp)_{-1}(0) = (a^\sharp a)_{-1}(0) = (a^\sharp)_{-1}(0), \\ \mathcal{R} &= a\mathcal{R} \oplus a^{-1}(0) = \mathcal{R}a \oplus a_{-1}(0). \end{aligned}$$

In the Banach context, other generalized inverses need to be recalled. To this end, however, some preparation is needed.

From now on, \mathcal{A} will denote a unitary Banach algebra with unit 1. Given $a \in \mathcal{A}$, $\sigma(a)$ will stand for the spectrum of a and when $\lambda \in \mathbb{C}$, $\lambda 1$ will be written simply as λ . In addition, \mathcal{X} will denote a Banach space and $L(\mathcal{X})$ the Banach algebra of all linear and bounded maps defined on and with values in \mathcal{X} . If $T \in L(\mathcal{X})$, then $N(T)$ and $R(T)$ will stand for the null space and the range of the operator T , respectively. In particular, when \mathcal{A} is a Banach algebra and $x \in \mathcal{A}$, the operators $L_x: \mathcal{A} \rightarrow \mathcal{A}$ and $R_x: \mathcal{A} \rightarrow \mathcal{A}$ are the maps defined as follows: given $z \in \mathcal{A}$, $L_x(z) = xz$ and $R_x(z) = zx$, respectively. Note that the identity operator defined on the Banach space \mathcal{X} will be denoted by $I \in L(\mathcal{X})$.

The following generalized inverse was introduced in the context of Banach algebras in [3]. Although the approach in this latter article is different from the one in [5], the generalized inverse introduced in [3] is related to the (b, c) -inverse (see section 7).

Definition 2.5 ([3, Definition 2.10]). *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. The element $y \in \mathcal{A}$ will be said to be the (p, q, l) -outer inverse of a , if the following identities hold:*

$$y = yay, \quad y\mathcal{A} = p\mathcal{A}, \quad y^{-1}(0) = q\mathcal{A}.$$

In this case, this outer inverse will be denoted by $a_{p, q}^{(2, l)}$.

Note that, in the same conditions of Definition 2.5, the (p, q, l) -outer inverse coincides with the hybrid $(p, 1 - q)$ -inverse.

One of the most studied generalized inverses is the $A_{T, S}^{(2)}$ outer inverse. This generalized inverse was studied for matrices and for operators defined on Hilbert and on Banach spaces. Since in this article Banach space operators will be considered, the definition of the aforementioned outer inverse will be given in the Banach frame. Recall that this inverse is unique, when it exists (see [10, Lemma 1]).

Definition 2.6. *Let \mathcal{X} be a Banach space, $A \in L(\mathcal{X})$ and T and S two closed subspaces of \mathcal{X} . If there exists a necessarily unique operator $B \in L(\mathcal{X})$ such that B is an outer inverse of A and $R(B) = T$ and $N(B) = S$, then B will be said to be the $A_{T, S}^{(2)}$ outer inverse of A .*

According to [10, Lemma 1], necessary and sufficient for the existence of the $A_{T, S}^{(2)}$ outer inverse of A is that T and S are complemented subspaces of \mathcal{X} , $A|_T^{A(T)}: T \rightarrow A(T)$

is invertible and $A(T) \oplus S = \mathcal{X}$. In particular, using this latter decomposition, $A_{T, S}^{(2)}$ is the following operator: $A_{T, S}^{(2)}|_S = 0$, $(A|_T^{A(T)})^{-1} = A_{T, S}^{(2)}|_{A(T)}^T: A(T) \rightarrow T$. To learn more properties of the $A_{T, S}^{(2)}$ outer inverse in Banach spaces, see [10, 12].

3. FURTHER EQUIVALENT CONDITIONS

In this section new conditions that are equivalent to the ones in Definition 2.1 will be given. First of all note that if \mathcal{R} is a unitary ring and $b, c, y \in \mathcal{R}$ are such that $y\mathcal{R} = b\mathcal{R}$, then $y_{-1}(0) = b_{-1}(0)$. Similarly from $\mathcal{R}y = \mathcal{R}c$ it can be derived that $y^{-1}(0) = c^{-1}(0)$. These results will be used in the following theorem.

Theorem 3.1. *Let \mathcal{R} be a unitary ring and consider $a \in \mathcal{R}$. Let $y \in \mathcal{R}$ be an outer inverse of a . Then, the following statements are equivalent.*

- (i) *The element y is the (b, c) -inverse of a .*
 - (ii) *$\mathcal{R}y = \mathcal{R}c$, $y\mathcal{R} \subseteq b\mathcal{R}$ and $y_{-1}(0) \subseteq b_{-1}(0)$.*
 - (iii) *$y\mathcal{R} = b\mathcal{R}$, $\mathcal{R}y \subseteq \mathcal{R}c$ and $y^{-1}(0) \subseteq c^{-1}(0)$.*
 - (iv) *$\mathcal{R}y \subseteq \mathcal{R}c$, $y\mathcal{R} \subseteq b\mathcal{R}$, $y_{-1}(0) \subseteq b_{-1}(0)$ and $y^{-1}(0) \subseteq c^{-1}(0)$.*
- In addition, if $b, c \in \hat{\mathcal{R}}$, then the following statements are equivalent to statement (i).
- (v) *$\mathcal{R}y = \mathcal{R}c$, $b\mathcal{R} \subseteq y\mathcal{R}$ and $b_{-1}(0) \subseteq y_{-1}(0)$.*
 - (vi) *$y\mathcal{R} = b\mathcal{R}$, $\mathcal{R}c \subseteq \mathcal{R}y$ and $c^{-1}(0) \subseteq y^{-1}(0)$.*
 - (vii) *$\mathcal{R}y = \mathcal{R}c$, $y_{-1}(0) = b_{-1}(0)$.*
 - (viii) *$\mathcal{R}y \subseteq \mathcal{R}c$, $b\mathcal{R} \subseteq y\mathcal{R}$, $y^{-1}(0) \subseteq c^{-1}(0)$ and $b_{-1}(0) \subseteq y_{-1}(0)$.*
 - (ix) *$\mathcal{R}c \subseteq \mathcal{R}y$, $y\mathcal{R} \subseteq b\mathcal{R}$, $y_{-1}(0) \subseteq b_{-1}(0)$ and $c^{-1}(0) \subseteq y^{-1}(0)$.*
 - (x) *$\mathcal{R}c \subseteq \mathcal{R}y$, $b\mathcal{R} \subseteq y\mathcal{R}$, $c^{-1}(0) \subseteq y^{-1}(0)$ and $b_{-1}(0) \subseteq y_{-1}(0)$.*
 - (xi) *$y\mathcal{R} = b\mathcal{R}$, $y^{-1}(0) = c^{-1}(0)$.*
 - (xii) *$\mathcal{R}y \subseteq \mathcal{R}c$, $y^{-1}(0) \subseteq c^{-1}(0)$ and $y_{-1}(0) = b_{-1}(0)$.*
 - (xiii) *$\mathcal{R}c \subseteq \mathcal{R}y$, $c^{-1}(0) \subseteq y^{-1}(0)$ and $y_{-1}(0) = b_{-1}(0)$.*
 - (xiv) *$b\mathcal{R} \subseteq y\mathcal{R}$, $b_{-1}(0) \subseteq y_{-1}(0)$ and $y^{-1}(0) = c^{-1}(0)$.*
 - (xv) *$y\mathcal{R} \subseteq b\mathcal{R}$, $y_{-1}(0) \subseteq b_{-1}(0)$ and $y^{-1}(0) = c^{-1}(0)$.*
 - (xvi) *$y^{-1}(0) = c^{-1}(0)$ and $y_{-1}(0) = b_{-1}(0)$.*

Proof. According to [5, Proposition 6.1] and the observation at the beginning of this section, it is clear that statement (i) implies all the other statements.

On the other hand, to prove that statements (ii)-(iv) imply that y is the (b, c) -inverse of a , note that according to [1, Proposition 3.1 (i)] (respectively [1, Proposition 3.1 (ii)]), if $y^{-1}(0) \subseteq c^{-1}(0)$ (respectively if $y_{-1}(0) \subseteq b_{-1}(0)$), then $\mathcal{R}c \subseteq \mathcal{R}y$ (respectively $b\mathcal{R} \subseteq y\mathcal{R}$).

Suppose that b and c are regular. Then, according to [1, Proposition 3.1 (iii)] (respectively [1, Proposition 3.1 (iv)]), if $c^{-1}(0) \subseteq y^{-1}(0)$ (respectively if $b_{-1}(0) \subseteq y_{-1}(0)$), then $\mathcal{R}y \subseteq \mathcal{R}c$ (respectively $y\mathcal{R} \subseteq b\mathcal{R}$). To conclude the proof, apply these results and the ones used to prove the equivalence among statements (i)-(iv). \square

Note that statements (xi) and (xx) in Theorem 3.1 correspond to the conditions of the hybrid and the annihilator (b, c) -inverse of a , respectively. The following corollary can be easily deduced from the previous theorem.

Corollary 3.2. *Let \mathcal{R} be a unitary ring and consider $a \in \mathcal{R}$ and $b, c \in \hat{\mathcal{R}}$. Then, the following statements are equivalent.*

- (i) *The element a has a (b, c) -inverse.*
- (ii) *The element a has an annihilator (b, c) -inverse.*
- (iii) *The element a has a hybrid (b, c) -inverse.*

Furthermore, in this case the inverses in statements (i)-(iii) coincide.

Proof. First apply the equivalence among statements (i), (xi) and (xx) in Theorem 3.1 and then apply [5, Theorem 2.1] and [5, Theorem 6.4]. \square

Let \mathcal{R} be a unitary ring and consider $a, b \in \mathcal{R}$ and $c \in \hat{\mathcal{R}}$ such that the hybrid (b, c) -inverse of a exists. Note that according to the proof of Theorem 3.1, the (b, c) -inverse of a exists and it coincides with the hybrid (b, c) -inverse of a . In particular, according to Remark 2.2 (i), $b \in \hat{\mathcal{R}}$.

4. THE SET OF (b, c) -INVERTIBLE ELEMENTS

The main objective of this section is to study the set of all (b, c) -invertible elements. In first place a characterization of the (b, c) -inverses will be given.

Theorem 4.1. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Consider $a \in \mathcal{R}$, $g \in b\{1\}$ and $h \in c\{1\}$. Then, the following conditions are equivalent.*

- (i) *The element a is (b, c) -invertible.*
- (ii) *There exists $z \in b\mathcal{R}c$ such that*

$$bg = zhcabg, \quad hc = hcabgz.$$

Furthermore, in this case $z = a^{-(b, c)}$.

Proof. Suppose that $a^{-(b, c)}$ exists. According to Definition 2.1 (i), $a^{-(b, c)} \in b\mathcal{R}c$. Next note that $b\mathcal{R}c = bg\mathcal{R}hc$. In particular, $a^{-(b, c)} = bga^{-(b, c)} = a^{-(b, c)}hc$. Thus, according to Definition 2.1 (ii),

$$bg = a^{-(b, c)}abg = a^{-(b, c)}hcabg, \quad hc = hcaa^{-(b, c)} = hcabga^{-(b, c)}.$$

To prove the converse, note that as before, since $z \in b\mathcal{R}c = bg\mathcal{R}hc$, $z = bgz = zhc$. In particular, $z \in b\mathcal{R}z$ and $z \in z\mathcal{R}c$. In addition,

$$b = bgb = zhcabgb = zab, \quad c = chc = hcabgz = caz.$$

Therefore, z satisfies the conditions of Definition 2.1. \square

From Theorem 4.1 new (b, c) -invertible elements can be constructed. This will be done in the following corollaries.

Corollary 4.2. *Let \mathcal{R} be a ring with unity and consider $a \in \mathcal{R}$, $b, c \in \hat{\mathcal{R}}$, $g \in b\{1\}$ and $h \in c\{1\}$. Then, the following statements are equivalent.*

- (i) *a is (b, c) -invertible.*
- (ii) *hca is (b, c) -invertible.*
- (iii) *abg is (b, c) -invertible.*
- (iv) *$hcabg$ is (b, c) -invertible.*

Furthermore, in this case,

$$a^{-(b, c)} = (hca)^{-(b, c)} = (abg)^{-(b, c)} = (hcabg)^{-(b, c)}.$$

Proof. Note that

$$hcabg = hc(hca)bg = hc(abg)bg = hc(hcabg)bg.$$

Now, necessary and sufficient for a to be (b, c) -invertible is that there exists $z \in \mathcal{R}$ such that Theorem 4.1 (ii) holds. However, this condition is equivalent to the existence of the outer inverses in statements (ii)-(iv). Moreover, in this case, all the outer inverses considered in statements (i)-(iv) coincide with z . \square

Corollary 4.3. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Let $a \in \mathcal{R}$ be such that $a^{-(b, c)}$ exists and consider $m \in hc\mathcal{R}(1 - bg) + (1 - hc)\mathcal{R}$, where $g \in b\{1\}$ and $h \in c\{1\}$. Then, $a + m$ is (b, c) -invertible. Moreover, $(a + m)^{-(b, c)} = a^{-(b, c)}$.*

Proof. Note that $hc(a + m)bg = hcabg$. Then, as in the proof of Corollary 4.2, there exists $z \in \mathcal{R}$ that satisfies Theorem 4.1 (ii). Now, this condition is also equivalent to the (b, c) -invertibility of $a + m$. Furthermore, both outer inverses under consideration coincide. \square

Corollary 4.4. *Let \mathcal{R} be a ring with unity and consider $a \in \mathcal{R}$ and $b, c \in \hat{\mathcal{R}}$ such that a is (b, c) -invertible. Let $g \in b\{1\}$ and $h \in c\{1\}$. Then, the following statements hold.*

- (i) $a^{-(b, c)} + m$ is (hc, bg) -invertible, where $m \in bg\mathcal{R}(1 - hc) + (1 - bg)\mathcal{R}$.
- (ii) $(a^{-(b, c)} + m)^{-(hc, bg)} = hcabg$; in particular, $(a^{-(b, c)})^{-(hc, bg)} = hcabg$.

Proof. Note that since hc and bg are idempotents, $hc \in hc\{1\}$ and $bg \in bg\{1\}$. In addition, since $a^{-(b, c)} \in b\mathcal{R}c = bg\mathcal{R}hc$ (Theorem 4.1), $bg(a^{-(b, c)} + m)hc = a^{-(b, c)}$. In particular, applying Theorem 4.1, the (hc, bg) -inverse of $a^{-(b, c)} + m$ exists and it coincides with $hcabg$. \square

In the following theorem the set of all (b, c) -invertible elements will be characterized ($b, c \in \hat{\mathcal{R}}$). To this end, however, first two sets need to be introduced. Let

$$\mathcal{R}^{-(b, c)} := \{x \in \mathcal{R} : x^{-(b, c)} \text{ exists} \}$$

and

$$\mathcal{R}_{\{hc, bg\}}^{-1} := \{x \in hc\mathcal{R}bg : \text{there exists } z \in b\mathcal{R}c \text{ such that } bg = zx, hc = xz\},$$

where $g \in b\{1\}$ and $h \in c\{1\}$ are arbitrary elements.

Theorem 4.5. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Let $g \in b\{1\}$ and $h \in c\{1\}$. Then,*

$$\mathcal{R}^{-(b, c)} = \mathcal{R}_{\{hc, bg\}}^{-1} + hc\mathcal{R}(1 - bg) + (1 - hc)\mathcal{R}.$$

Furthermore, if $x \in \mathcal{R}_{\{hc, bg\}}^{-1}$ and $m \in hc\mathcal{R}(1 - bg) + (1 - hc)\mathcal{R}$, then $(x + m)^{-(b, c)} = x^{-(b, c)} \in b\mathcal{R}c$.

Proof. Let $x \in \mathcal{R}_{\{hc, bg\}}^{-1}$ and consider $z \in b\mathcal{R}c$ such that $bg = zx$ and $hc = xz$. According to Theorem 4.1, x is (b, c) -invertible. In addition, $x^{-(b, c)} = z$. Now let $m \in hc\mathcal{R}(1 - bg) + (1 - hc)\mathcal{R}$. According to Corollary 4.3, $x + m$ is (b, c) -invertible and $(x + m)^{-(b, c)} = x^{-(b, c)} = z$.

On the other hand, let $a \in \mathcal{R}^{-(b, c)}$. According to Theorem 4.1, $hcabg = x \in \mathcal{R}_{\{hc, bg\}}^{-1}$. Note that $a = hcabg + m$, where $m = hca(1 - bg) + (1 - hc)a \in hc\mathcal{R}(1 - bg) + (1 - hc)\mathcal{R}$. \square

From Theorem 4.5 a new characterization of (b, c) -invertible elements can be derived.

Corollary 4.6. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Let $a \in \mathcal{R}$ and consider $g \in b\{1\}$ and $h \in c\{1\}$. Then, the following statements are equivalent.*

- (i) *The element a is (b, c) -invertible.*
- (ii) *There exist $u, v \in \mathcal{R}$ such that $u \in \mathcal{R}_{\{hc, bg\}}^{-1}$, $hcvbg = 0$ and $a = u + v$.*

Proof. Apply Theorem 4.5. \square

To end this section, the relationship between $\mathcal{R}_{\{hc, bg\}}^{-1}$ and the product will be studied. Recall however first that given a ring with unity \mathcal{R} and $p \in \mathcal{R}^\bullet$, $p\mathcal{R}p$ is a ring with unit p . Moreover, if $t \in (p\mathcal{R}p)^{-1}$, then its inverse in $p\mathcal{R}p$ will be denoted by $t_{p\mathcal{R}p}^{-1}$. In particular, $tt_{p\mathcal{R}p}^{-1} = t_{p\mathcal{R}p}^{-1}t = p$.

Theorem 4.7. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Let $g \in b\{1\}$ and $h \in c\{1\}$. The following statements hold.*

- (i) $\mathcal{R}_{\{hc, bg\}}^{-1} = (hc\mathcal{R}hc)^{-1}\mathcal{R}_{\{hc, bg\}}^{-1}(bg\mathcal{R}bg)^{-1}$.
- (ii) *Let $x \in \mathcal{R}_{\{hc, bg\}}^{-1}$ and consider $u \in (bg\mathcal{R}bg)^{-1}$ and $v \in (hc\mathcal{R}hc)^{-1}$. Let $m \in hc\mathcal{R}(1 - bg) + (1 - hc)\mathcal{R}$. Then, $vxu + m \in \mathcal{R}^{-(b, c)}$ and*

$$(vxu)^{-(b, c)} = (vxu + m)^{-(b, c)} = u_{bg\mathcal{R}bg}^{-1}x^{-(b, c)}v_{hc\mathcal{R}hc}^{-1}.$$

Proof. Note that since $\mathcal{R}_{\{hc, bg\}}^{-1} \subseteq hc\mathcal{R}bg$,

$$\mathcal{R}_{\{hc, bg\}}^{-1} = hc\mathcal{R}_{\{hc, bg\}}^{-1}bg \subseteq (hc\mathcal{R}hc)^{-1}\mathcal{R}_{\{hc, bg\}}^{-1}(bg\mathcal{R}bg)^{-1}.$$

To prove the other inclusion, let $x \in \mathcal{R}_{\{hc, bg\}}^{-1}$. Since $hcxbg = x$, according to Theorem 4.1, there exists $z \in b\mathcal{R}c = bg\mathcal{R}hc$ such that $zx = bg$ and $xz = hc$. Consider $u \in (bg\mathcal{R}bg)^{-1}$ and $v \in (hc\mathcal{R}hc)^{-1}$ and set $x' = vxu$ and $z' = u_{bg\mathcal{R}bg}^{-1}x^{-(b, c)}v_{hc\mathcal{R}hc}^{-1}$. Note that $x' \in hc\mathcal{R}bg$ and $z' \in bg\mathcal{R}hc = b\mathcal{R}c$. Moreover, direct calculations proves that $z'x' = bg$ and $x'z' = hc$. In particular, the first statement holds.

To conclude the proof, apply Corollary 4.3. \square

5. THE BOTT-DUFFIN INVERSE

In this section the relationship between the (b, c) -inverse and the Bott-Duffin (p, q) -inverse will be studied. Recall that the Bott-Duffin inverse is a particular case of the (b, c) -inverse. The next result will show that these definitions are essentially the same. In fact, any (b, c) -inverse can be presented as a Bott-Duffin inverse.

Proposition 5.1. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Let $a \in \mathcal{R}$ and consider $g \in b\{1\}$ and $h \in c\{1\}$. The following statements are equivalent.*

- (i) *The element a is (b, c) -invertible.*
- (ii) *The element a is Bott-Duffin (bg, hc) -invertible.*

Furthermore, in this case

$$a^{-(b, c)} = a^{-(bg, hc)} \text{ and } \mathcal{R}^{-(b, c)} = \mathcal{R}^{-(bg, hc)}.$$

Proof. Apply Theorem 4.1. □

Next the relationship between the Bott-Duffin inverse and the usual inverse will be studied in a particular case. To this end, recall first the following well known result. Let \mathcal{R} be a ring with unity and consider $p \in \mathcal{R}^\bullet$ and $a \in \mathcal{R}$ such that $ap = pa$ (equivalently $a = pap + (1 - p)a(1 - p)$). Then, necessary and sufficient for $a \in \mathcal{R}^{-1}$ is that $pap \in (p\mathcal{R}p)^{-1}$ and $(1 - p)a(1 - p) \in ((1 - p)\mathcal{R}(1 - p))^{-1}$. Moreover, in this case, a direct calculation proves that if $a_1 \in p\mathcal{R}p$ (respectively $a_2 \in (1 - p)\mathcal{R}(1 - p)$) is the inverse of pap in $p\mathcal{R}p$ (respectively the inverse of $(1 - p)a(1 - p)$ in $(1 - p)\mathcal{R}(1 - p)$), then $a^{-1} = a_1 + a_2$. In the following lemma, this result will be extended to the case of two idempotents.

Lemma 5.2. *Let \mathcal{R} be a ring with unity and consider $p, q \in \mathcal{R}^\bullet$. Suppose that $a \in \mathcal{R}$ is such that $ap = qa$. Then the following statements are equivalent.*

- (i) *The element a is invertible.*
- (ii) *There is $z \in \mathcal{R}$ such that $zq = pz$ and*

$$\begin{aligned} pzqap &= p, & (1 - p)z(1 - q)a(1 - p) &= 1 - p, \\ gapzq &= q, & (1 - q)a(1 - p)z(1 - q) &= 1 - q. \end{aligned}$$

Furthermore, in this case $z = a^{-1}$.

Proof. Note that the condition $ap = qa$ is equivalent to $a = qap + (1 - q)a(1 - p)$, which in turn is equivalent to $(1 - q)ap = 0 = qa(1 - p)$. Similarly, given $z \in \mathcal{R}$, necessary and sufficient for $zq = pz$ is that $z = pzq + (1 - p)z(1 - q)$, equivalently, $(1 - p)zq = 0 = pz(1 - q)$.

The statement of the Lemma can be proved by direct computation. The details are left to the reader. □

Next a relation between the Bott-Duffin inverse and the usual inverse will be established.

Theorem 5.3. *Let \mathcal{R} be a ring with unity and consider $p, q \in \mathcal{R}^\bullet$. Let $a \in \mathcal{R}$ be such that $ap = qa$. Then, the following statements are equivalent.*

- (i) *The element $a \in \mathcal{R}^{-1}$.*
- (ii) *Both the Bott-Duffin (p, q) -inverse of a and the Bott-Duffin $(1 - p, 1 - q)$ -inverse of a exist.*

Furthermore, in this case $a^{-1} = a^{-(p, q)} + a^{-(1-p, 1-q)}$.

Proof. Since $ap = qa$, $a = a_1 + a_2$, where $a_1 = qap \in q\mathcal{R}p$ and $a_2 = (1 - q)a(1 - p) \in (1 - q)\mathcal{R}(1 - p)$. According to Lemma 5.2, necessary and sufficient for $a \in \mathcal{R}^{-1}$ is that

there exists $z = z_1 + z_2$, $z_1 = pzq \in p\mathcal{R}q$ and $z_2 = (1-p)z(1-q) \in (1-p)\mathcal{R}(1-q)$, such that

$$\begin{aligned} z_1 a_1 &= p, & a_1 z_1 &= q, \\ z_2 a_2 &= 1-p, & a_2 z_2 &= 1-q. \end{aligned}$$

Moreover, in this case $z = a^{-1}$. However, according to Theorem 4.1, these equations are equivalent to the fact that $a^{-(p, q)}$ and $a^{-(1-p, 1-q)}$ exist.

To prove the converse, let $z = a^{-(p, q)} + a^{-(1-p, 1-q)}$. Clearly, $zq = pz$. In addition, since $pzq = a^{-(p, q)}$ and $(1-p)z(1-q) = a^{-(1-p, 1-q)}$, according to Lemma 5.2, a is invertible. \square

In the following corollary the relationship between the (b, c) -inverse and the usual inverse will be studied.

Corollary 5.4. *Let \mathcal{R} be a ring with unity and consider $b, c \in \hat{\mathcal{R}}$. Let $g \in b\{1\}$ and $h \in c\{1\}$. Then, if $a \in \mathcal{R}$ is such that $abg = hca$, the following statements are equivalent.*

- (i) *The element $a \in \mathcal{R}^{-1}$.*
- (ii) *a is (b, c) -invertible and the Bott-Duffin $(1-bg, 1-hc)$ -inverse of a exists. Furthermore, in this case $a^{-1} = a^{-(b, c)} + a^{-(1-bg, 1-hc)}$.*

Proof. Define $p = bg$ and $q = hc$. Apply then Theorem 5.3 and Proposition 5.1. \square

6. THE REVERSE ORDER LAW

The main objective of this section is to study the reverse order law. In first place the notion of reverse order law for the (b, c) -inverse will be introduced.

Definition 6.1. *Let \mathcal{R} be a ring with unity and consider $b_i, c_i \in \hat{\mathcal{R}}$ ($i = 1, 2$). Let $g_i \in b_i\{1\}$ and $h_i \in c_i\{1\}$ ($i = 1, 2$) and suppose that $h_2 c_2 = b_1 g_1$. Let $a_i \in \mathcal{R}$ and suppose that $a^{-(b_i, c_i)}$ exist ($i = 1, 2$). It will be said that $a_1 a_2$ satisfies the reverse order law, if $a_1 a_2$ is (b_2, c_1) -invertible and*

$$(a_1 a_2)^{-(b_2, c_1)} = a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)}.$$

Next the reverse order law will be studied.

Theorem 6.2. *Let \mathcal{R} be a ring with unity and consider $b_i, c_i \in \hat{\mathcal{R}}$ ($i = 1, 2$). Let $g_i \in b_i\{1\}$ and $h_i \in c_i\{1\}$ ($i = 1, 2$) and suppose that $h_2 c_2 = b_1 g_1$. Let $a_i \in \mathcal{R}$ and suppose that $a^{-(b_i, c_i)}$ exist ($i = 1, 2$). Then, the following statements are equivalent.*

- (i) $h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2 = 0$.
- (ii) *The element $a_1 a_2$ satisfies the reverse order law.*

Proof. Let $a_i^{-(b_i, c_i)} \in b_i \mathcal{R} c_i$ ($i = 1, 2$). Recall that according to Theorem 4.1, for $i = 1, 2$,

$$b_i g_i = a_i^{-(b_i, c_i)} h_i c_i a_i b_i g_i, \quad h_i c_i = h_i c_i a_i b_i g_i a_i^{-(b_i, c_i)}.$$

Since $h_2 c_2 = b_1 g_1$, it is not difficult to prove that

$$h_1 c_1 a_1 a_2 b_2 g_2 = h_1 c_1 a_1 b_1 g_1 a_2 b_2 g_2 + h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2.$$

Suppose that $h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2 = 0$. Thus, $a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} \in b_2 \mathcal{R} c_1$ and, according to Theorem 4.1,

$$\begin{aligned} a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 a_2 b_2 g_2 &= a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 b_1 g_1 h_2 c_2 a_2 b_2 g_2 \\ &= a_2^{-(b_2, c_2)} b_1 g_1 h_2 c_2 a_2 b_2 g_2 = a_2^{-(b_2, c_2)} h_2 c_2 a_2 b_2 g_2 \\ &= b_2 g_2, \end{aligned}$$

and

$$\begin{aligned} h_1 c_1 a_1 a_2 b_2 g_2 a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} &= h_1 c_1 a_1 b_1 g_1 h_2 c_2 a_2 b_2 g_2 a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} \\ &= h_1 c_1 a_1 b_1 g_1 h_2 c_2 a_1^{-(b_1, c_1)} = h_1 c_1 a_1 b_1 g_1 a_1^{-(b_1, c_1)} \\ &= h_1 c_1. \end{aligned}$$

Therefore, according to Theorem 4.1, $a_1 a_2$ is (b_2, c_1) -invertible and $a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} = (a_1 a_2)^{-(b_2, c_1)}$.

To prove the converse, suppose that $a_1 a_2$ is (b_2, c_1) -invertible and $a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} = (a_1 a_2)^{-(b_2, c_1)}$. In particular, according to Theorem 4.1,

$$\begin{aligned} b_2 g_2 &= a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 a_2 b_2 g_2 \\ &= a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 b_1 g_1 a_2 b_2 g_2 + a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2 \\ &= b_2 g_2 + a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2. \end{aligned}$$

As a result,

$$a_2^{-(b_2, c_2)} a_1^{-(b_1, c_1)} h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2 = 0.$$

Now, multiplying the above equation on the left side by $h_2 c_2 a_2 b_2 g_2$,

$$0 = h_2 c_2 a_1^{-(b_1, c_1)} h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2 = a_1^{-(b_1, c_1)} h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2.$$

Finally, multiplying again the above equation on the left side by $h_1 c_1 a_1 b_1 g_1$,

$$h_1 c_1 a_1 (1 - b_1 g_1) a_2 b_2 g_2 = 0.$$

□

Since the Bott-Duffin inverse is a particular case of the (b, c) -inverse, the following corollary can be directly derived from Theorem 6.2.

Corollary 6.3. *Let \mathcal{R} be a ring with unity and consider $p, q, r \in \mathcal{R}^\bullet$. Let $a_i \in \mathcal{R}$, $i = 1, 2$, and suppose that a_1 is Bott-Duffin (p, q) -invertible and a_2 is Bott-Duffin (r, p) -Bott Duffin invertible. Then, the following statements are equivalent.*

- (i) $q a_1 (1 - p) a_2 r = 0$.
- (ii) *The element $a_1 a_2$ is Bott-Duffin (r, q) -invertible and*

$$(a_1 a_2)^{-(r, q)} = a_2^{-(r, p)} a_1^{-(p, q)}.$$

Proof. Apply Theorem 6.2 and use p (respectively q, r) as a generalized inverse of the idempotent p (respectively q, r). □

7. PROPERTIES OF THE (b, c) -INVERSE IN BANACH ALGEBRAS

In this section several results concerning representations, limits, approximation and continuity of the (b, c) -inverse will be studied in the context of Banach algebras. In first place a lemma will be presented.

Lemma 7.1. *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$. Let $g \in b\{1\}$ and $h \in c\{1\}$. Then, the following statements are equivalent.*

(i) *The element a is (b, c) -invertible.*

(ii) *The outer inverse $a_{bg, 1-hc}^{(2, l)}$ exists.*

Furthermore, in this case $a^{-(b, c)}$ and $a_{bg, 1-hc}^{(2, l)}$ coincide.

Proof. Recall that according to Proposition 5.1, statement (i) is equivalent to the existence of the Bott-Duffin (bg, hc) -inverse of a , moreover, this two inverses coincide, i.e., $a^{-(b, c)} = a^{-(bg, hc)}$. In addition, according to Corollary 3.2, statement (i) is equivalent to the existence of the hybrid (bg, hc) -inverse of a and this latter inverse coincides with $a^{-(b, c)} = a^{-(bg, hc)}$. However, according to Definition 2.5, the hybrid (bg, hc) -inverse of a coincides with $a_{bg, 1-hc}^{(2, l)}$. \square

In order to prove the main results of this section, some facts need to be recalled first.

Remark 7.2. Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists.

(i) Since $a^{-(b, c)} \in b\mathcal{A}c$ (Theorem 4.1), $a^{-(b, c)}\mathcal{A} \subseteq b\mathcal{A}$ and $c^{-1}(0) \subseteq (a^{-(b, c)})^{-1}(0)$. On the other hand, since according again to Theorem 4.1

$$bg = a^{-(b, c)}hcabg, \quad hc = hcabga^{-(b, c)},$$

($g \in b\{1\}$ and $h \in c\{1\}$), $b\mathcal{A} \subseteq a^{-(b, c)}\mathcal{A}$ and $(a^{-(b, c)})^{-1}(0) \subseteq c^{-1}(0)$. Thus, $a^{-(b, c)}\mathcal{A} = b\mathcal{A}$ and $(a^{-(b, c)})^{-1}(0) = c^{-1}(0)$.

Let \mathcal{A}_{op} be the Banach algebra, which as Banach space coincides with \mathcal{A} , but whose product is the opposite of the one in \mathcal{A} , i.e., $x \cdot_{op} y = yx$ ($x, y \in \mathcal{A}$).

(ii) Note that the (b, c) -inverse of $a \in \mathcal{A}$ exists if and only if the (c, b) -inverse of $a \in \mathcal{A}_{op}$ exists. Moreover, in this case both inverses coincide.

(iii) Let $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$. Then, according to Lemma 7.1 and [3, Theorem 4.1], av and va are group invertible and $a^{-(b, c)} = (va)^{\#}v = v(av)^{\#}$. In particular,

$$va\mathcal{A} = v\mathcal{A} = b\mathcal{A} = bg\mathcal{A}, \quad (av)^{-1}(0) = v^{-1}(0) = c^{-1}(0) = (hc)^{-1}(0).$$

Now an easy calculation proves the following fact: given $p, q \in \mathcal{A}^{\bullet}$, $p\mathcal{A} = q\mathcal{A}$ in \mathcal{A} if and only if $p^{-1}(0) = q^{-1}(0)$ in \mathcal{A}_{op} . Consequently, applying this result to $p = va$ and $q = bg$, it is not difficult to prove that the identity $v\mathcal{A} = b\mathcal{A}$ in \mathcal{A} is equivalent to the identity $v^{-1}(0) = b^{-1}(0)$ in \mathcal{A}_{op} .

Similarly, the following statement holds: $p^{-1}(0) = q^{-1}(0)$ in \mathcal{A} if and only if $p\mathcal{A} = q\mathcal{A}$ in \mathcal{A}_{op} . Therefore, applying this latter statement to $p = av$ and $q = hc$, it is not difficult to prove that $v^{-1}(0) = c^{-1}(0)$ in \mathcal{A} is equivalent to $v\mathcal{A} = c\mathcal{A}$ in \mathcal{A}_{op} .

In the following theorem an integral representation of the (b, c) -inverse will be derived from the corresponding representation of the outer inverse $a_{p, q}^{(2, l)}$.

Theorem 7.3. *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists. Consider $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$ and suppose that for every $\lambda \in \sigma(av)$, $\operatorname{Re}(\lambda) \geq 0$. Then*

$$a^{-(b, c)} = \int_0^\infty v e^{-(av)t} dt = \int_0^\infty e^{-(va)t} v dt.$$

Proof. Let $g \in b\{1\}$ and $h \in c\{1\}$. According to Lemma 7.1, $a_{bg, 1-hc}^{(2, l)}$ exists and $a^{-(b, c)} = a_{bg, 1-hc}^{(2, l)}$. Next note that $bg\mathcal{A} = b\mathcal{A}$ and $c^{-1}(0) = (1 - hc)\mathcal{A}$. Then apply [3, Theorem 4.9] to deduce that

$$a^{-(b, c)} = \int_0^\infty v e^{-(av)t} dt.$$

To prove the second equality, first note that according to statement (ii) of Remark 7.2, the (c, b) -inverse of $a \in \mathcal{A}_{op}$ exists and coincide with $a^{-(b, c)}$. In addition, according to statement (iii) of Remark 7.2, $v \in \mathcal{A}_{op}$ is such that $v \cdot_{op} \mathcal{A} = c \cdot_{op} \mathcal{A}$ and $v^{-1}(0) = b^{-1}(0)$ (in \mathcal{A}_{op}). Moreover, since $\sigma(a \cdot_{op} v) = \sigma(va)$ and $\sigma(va) \setminus \{0\} = \sigma(av) \setminus \{0\}$ ([2, Chapter 1, Section 5, Proposition 3]), $\lambda \in \sigma(av)$ is such that $\operatorname{Re}(\lambda) \geq 0$ if and only if $\lambda \in \sigma(a \cdot_{op} v) = \sigma(va)$ is such that $\operatorname{Re}(\lambda) \geq 0$. Therefore, the second integral representation can be easily deduced from the identity that has already been proved. \square

Next the (b, c) -inverse will be represented by means of a convergent series. However, to this end, it is necessary to recall some facts.

Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$. According to [3, Theorem 4.1] and Lemma 7.1, if $a^{-(b, c)}$ exists and $v \in \mathcal{A}$ is such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$, then va is group invertible and $a^{-(b, c)} = (va)^\# v$. Set $p_{va} = va(va)^\#$. Note that since va is group invertible, then $va = p_{va}va = vap_{va} = p_{va}vap_{va}$. In addition, $va \in (p_{va}\mathcal{A}p_{va})^{-1}$ and the inverse of va in $p_{va}\mathcal{A}p_{va}$ is $(va)^\# \in p_{va}\mathcal{A}p_{va}$ ($p_{va}\mathcal{A}p_{va}$ is a Banach algebra with unit p_{va}). Recall also that according to [3, Theorem 4.1] and Lemma 7.1, av is group invertible, $a^{-(b, c)} = v(av)^\#$, and if $p_{av} = av(av)^\# \in \mathcal{A}^\bullet$, then $av = p_{av}av = avp_{av} = p_{av}avp_{av}$.

Theorem 7.4. *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists. Let $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$ and consider $\beta \in \mathbb{C} \setminus \{0\}$ such that $\|p_{va} - \beta va\| < 1$. Then*

$$a^{-(b, c)} = \beta \sum_{n=0}^{\infty} (1 - \beta va)^n v = \beta \sum_{n=0}^{\infty} v (1 - \beta av)^n.$$

Proof. Since $\|p_{va} - \beta va\| < 1$,

$$(va)^\# = \beta \sum_{n=0}^{\infty} (p_{va} - \beta va)^n.$$

On the other hand, since $(1 - \beta va) = p_{va} - \beta va + 1 - p_{va}$, it is not difficult to prove that $(1 - \beta va)^n = (p_{va} - \beta va)^n + 1 - p_{va}$ ($n \in \mathbb{N}$). Thus, given $k \in \mathbb{N}$,

$$\sum_{n=0}^k (1 - \beta va)^n = \sum_{n=0}^k (p_{va} - \beta va)^n + (k+1)(1 - p_{va}).$$

Now let $g \in b\{1\}$ and $h \in c\{1\}$. Note that according to [3, Theorem 4.1] and Theorem 4.1

$$\begin{aligned} p_{va} &= (va)^{\sharp} va = a^{-(b, c)} a = a^{-(b, c)} h cabg + a^{-(b, c)} h ca(1 - bg) \\ &= bg + a^{-(b, c)} h ca(1 - bg). \end{aligned}$$

Since $v \in b\mathcal{A} = bg\mathcal{A}$, there is $z \in \mathcal{A}$ such that $v = bgz$. Thus,

$$p_{va}v = bgv + a^{-(b, c)} h ca(1 - bg)v = v,$$

equivalently, $(1 - p_{va})v = 0$.

Consequently, given $k \in \mathbb{N}$,

$$\beta \sum_{n=0}^k (1 - \beta va)^n v = \beta \left(\sum_{n=0}^k (1 - \beta va)^n \right) v = \beta \sum_{n=0}^k (p_{va} - \beta va)^n v.$$

Therefore, according to [3, Theorem 4.1],

$$\beta \sum_{n=0}^{\infty} (1 - \beta va)^n v = \beta \sum_{n=0}^{\infty} (p_{va} - \beta va)^n v = (va)^{\sharp} v = a^{-(b, c)}.$$

To prove that $a^{-(b, c)}$ coincides with the second serie, proceed as in the proof of Theorem 7.3. In particular, recall that the (c, b) -inverse of $a \in \mathcal{A}_{op}$ exists and coincides with $a^{-(b, c)}$. Moreover, $v \in \mathcal{A}_{op}$ is such that $v \cdot_{op} \mathcal{A} = c \cdot_{op} \mathcal{A}$ and $v^{-1}(0) = b^{-1}(0)$ (in \mathcal{A}_{op}). Thus, to derive the proof of the second representation from what has been already proved, it is enough to show that $\|p_{av} - \beta av\| = \|p_{va} - \beta va\|$.

To this end, recall that since va is group invertible, $\mathcal{A} = p_{va}\mathcal{A} \oplus (1 - p_{va})\mathcal{A}$. Next consider the operator $L_{va}: \mathcal{A} \rightarrow \mathcal{A}$. Note that (i) $L_{va}|_{(1-p_{va})\mathcal{A}} = 0$, (ii) since $L_{va}|_{p_{va}\mathcal{A}}$ is invertible ($(L_{va}|_{p_{va}\mathcal{A}})^{-1} = L_{(va)^{\sharp}}|_{p_{va}\mathcal{A}}$), $0 \notin \sigma(L_{va}|_{p_{va}\mathcal{A}})$, (iii) since $\sigma(va) = \sigma(L_{va})$ ([2, Chapter 1, Section 5, Proposition 4 (ii)]), $\sigma(va) = \{0\} \cup \sigma(L_{va}|_{p_{va}\mathcal{A}})$.

On the other hand, according to [3, Theorem 4.1] and Lemma 7.1, av is group invertible. In particular, since $p_{av} = av(av)^{\sharp} \in \mathcal{A}^{\bullet}$, $\mathcal{A} = p_{av}\mathcal{A} \oplus (1 - p_{av})\mathcal{A}$. In addition, using similar arguments to the ones in the previous paragraph, it is not difficult to prove that $\sigma(av) = \{0\} \cup \sigma(L_{av}|_{p_{av}\mathcal{A}})$ and $0 \notin \sigma(L_{av}|_{p_{av}\mathcal{A}})$.

Now since $\sigma(av) \setminus \{0\} = \sigma(va) \setminus \{0\}$ ([2, Chapter 1, Section 5, Proposition 3]) and $0 \notin (\sigma(L_{av}|_{p_{av}\mathcal{A}}) \cup \sigma(L_{va}|_{p_{va}\mathcal{A}}))$, $\sigma(L_{av}|_{p_{av}\mathcal{A}}) = \sigma(L_{va}|_{p_{va}\mathcal{A}})$.

Moreover, since $L_{p_{va}-\beta va}(p_{va}\mathcal{A}) \subseteq p_{va}\mathcal{A}$ and $L_{p_{va}-\beta va}(1 - p_{va}\mathcal{A}) = 0$, $\sigma(L_{p_{va}-\beta va}) = \sigma(L_{p_{va}-\beta va}|_{p_{va}\mathcal{A}}) \cup \{0\}$. Note that, according to the Functional Calculus Theorem applied to the polynomial $P(X) = 1 - \beta X \in \mathbb{C}[X]$ and the operator $L_{p_{va}-\beta va}|_{p_{va}\mathcal{A}} = (I - \beta L_{va})|_{p_{va}\mathcal{A}}$, $\sigma(L_{p_{va}-\beta va}|_{p_{va}\mathcal{A}}) = 1 - \beta \sigma(L_{va}|_{p_{va}\mathcal{A}})$. However, using similar arguments

it is possible to prove the following facts: $\sigma(L_{p_{av}-\beta av}) = \sigma(L_{p_{av}-\beta av} |_{p_{av}\mathcal{A}}) \cup \{0\}$ and $\sigma(L_{p_{av}-\beta av} |_{p_{av}\mathcal{A}}) = 1 - \beta\sigma(L_{av} |_{p_{av}\mathcal{A}})$. As a result, $\sigma(L_{p_{va}-\beta va}) = \sigma(L_{p_{av}-\beta av})$.

Now, according to [2, Chapter 1, Section 5, Theorem 8] and [2, Chapter 1, Section 5, Proposition 4 (ii)],

$$\begin{aligned} \|p_{av} - \beta av\| &= \max\{|\lambda| : \lambda \in \sigma(p_{av} - \beta av)\} = \max\{|\lambda| : \lambda \in \sigma(L_{p_{av}-\beta av})\} \\ &= \max\{|\lambda| : \lambda \in \sigma(L_{p_{va}-\beta va})\} = \max\{|\lambda| : \lambda \in \sigma(p_{va} - \beta va)\} \\ &= \|p_{va} - \beta va\| \end{aligned}$$

□

Now due to [3, Theorem 4.7] the (b, c) -inverse will be presented as a limit.

Theorem 7.5. *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists. Let $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$. Then*

$$a^{-(b, c)} = \lim_{\substack{\lambda \xrightarrow{\sigma} 0 \\ \lambda \notin \sigma(-av)}} v(\lambda + av)^{-1} = \lim_{\substack{\lambda \xrightarrow{\sigma} 0 \\ \lambda \notin \sigma(-va)}} (\lambda + va)^{-1}v.$$

Proof. To prove the first identity, recall that according to Lemma 7.1, $a^{-(b, c)} = a_{bg, 1-hc}^{(2, l)}$. In addition, given $g \in b\{1\}$ and $h \in c\{1\}$, according to the hypotheses, $v\mathcal{A} = bg\mathcal{A}$ and $v^{-1}(0) = (1 - hc)\mathcal{A}$. Thus, according to [3, Theorem 4.1], av is group invertible. In particular, according to [9, Theorem 4], 0 is an isolated point of $\sigma(-av)$. Then, there is an open set $U \subset \mathbb{C}$ such that $0 \in U$ and $U \cap \sigma(-av) = \{0\}$. To conclude the proof, apply [3, Theorem 4.7].

In order to prove the remaining limit, proceed as in the proof of Theorem 7.3. In particular, recall that (c, b) -inverse of $a \in \mathcal{A}_{op}$ exists and coincide with $a^{-(b, c)}$. Moreover, $v \in \mathcal{A}_{op}$ is such that $v \cdot_{op} \mathcal{A} = c \cdot_{op} \mathcal{A}$ and $v^{-1}(0) = b^{-1}(0)$ (in \mathcal{A}_{op}). In addition, according to [2, Chapter 1, Section 5, Proposition 3], $\sigma(va) \setminus \{0\} = \sigma(av) \setminus \{0\}$. Therefore, the second identity can be derived from the first. □

Under the same hypothesis of Theorem 7.5, to establish a bound for $\|a^{-(b, c)} - v(\lambda + av)^{-1}\|$, it is necessary to note the following facts.

Remark 7.6. Let \mathcal{A} be a Banach algebra and consider $x, y \in \mathcal{A}$ two invertible elements. Then,

$$x^{-1} - y^{-1} = y^{-1}(y - x)x^{-1} = (y^{-1} - x^{-1})(y - x)x^{-1} + x^{-1}(y - x)x^{-1}.$$

Thus,

$$\|x^{-1} - y^{-1}\| \leq \|y^{-1} - x^{-1}\| \|y - x\| \|x^{-1}\| + \|x^{-1}\|^2 \|y - x\|.$$

In particular, when $\|y - x\| < \frac{1}{\|x^{-1}\|}$,

$$\|x^{-1} - y^{-1}\| \leq \frac{\|x^{-1}\|^2 \|y - x\|}{1 - \|y - x\| \|x^{-1}\|}.$$

Remark 7.7. Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists. Let $g \in b\{1\}$, $h \in c\{1\}$ and $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$. Since according to Lemma 7.1, $a^{-(b, c)} = a_{bg, 1-hc}^{(2, l)}$, due to [3, Theorem 3.2], $\mathcal{A} = ab\mathcal{A} \oplus c^{-1}(0)$ and $a^{-1}(0) \cap b\mathcal{A} = 0$ (since $b, c \in \hat{\mathcal{A}}$, $b\mathcal{A} = bg\mathcal{A}$ and $(1 - hc)\mathcal{A} = c^{-1}(0)$). In addition, according to the proof of [3, Theorem 4.1], $(av)^{-1}(0) = v^{-1}(0) = c^{-1}(0)$. Now, since av is group invertible ([3, Theorem 4.1]),

$$\mathcal{A} = av\mathcal{A} \oplus (av)^{-1}(0) = ab\mathcal{A} \oplus c^{-1}(0).$$

Note that $av\mathcal{A} = av(av)^\# \mathcal{A}$ and $(av)^{-1}(0) = (1 - av(av)^\#)\mathcal{A}$ are two closed and complemented subspace of \mathcal{A} .

Consider the maps $L_a, L_v : \mathcal{A} \rightarrow \mathcal{A}$. Since $a^{-1}(0) \cap v\mathcal{A} = 0$, $L_a|_{v\mathcal{A}}^{av\mathcal{A}} : v\mathcal{A} \rightarrow av\mathcal{A}$ is an isomorphism. Moreover, since $(av)^{-1}(0) = v^{-1}(0)$, $vav\mathcal{A} = v\mathcal{A}$ and $L_v|_{av\mathcal{A}}^{v\mathcal{A}} : av\mathcal{A} \rightarrow v\mathcal{A}$ is an isomorphism. Then $L_{av}|_{av\mathcal{A}} = L_a|_{v\mathcal{A}}^{av\mathcal{A}} L_v|_{av\mathcal{A}}^{v\mathcal{A}} : av\mathcal{A} \rightarrow av\mathcal{A}$ and

$$L_{(av)^\#}|_{av\mathcal{A}} = (L_{av}|_{av\mathcal{A}})^{-1} = (L_v|_{av\mathcal{A}}^{v\mathcal{A}})^{-1} (L_a|_{v\mathcal{A}}^{av\mathcal{A}})^{-1}$$

(note that if $v = 0$ or $a = 0$, then $a^{-(b, c)} = 0$ and $\|a^{-(b, c)} - v(\lambda + av)^{-1}\| = 0$, i.e., no bound needs to be studied). In addition, note that since $a^{-(b, c)} = a_{bg, 1-hc}^{(2, l)}$, $L_{a^{-(b, c)}} = L_{a_{bg, 1-hc}^{(2, l)}} = (L_a)_{b\mathcal{A}, c^{-1}(0)}^{(2)}$.

Now consider the operator $H \in \mathbb{L}(\mathcal{A})$ defined as follows:

$$H|_{(1-bg)\mathcal{A}} = 0, \quad H|_{v\mathcal{A}}^{av\mathcal{A}} = (L_v|_{av\mathcal{A}}^{v\mathcal{A}})^{-1} : v\mathcal{A} \rightarrow av\mathcal{A}$$

($bg\mathcal{A} = b\mathcal{A} = v\mathcal{A}$, $\mathcal{A} = bg\mathcal{A} \oplus (1 - bg)\mathcal{A}$). Then, $L_{(av)^\#} = HL_{a^{-(b, c)}}$.

In fact, $N(L_{(av)^\#}) = (av)^{-1}(0) = c^{-1}(0) = N((L_a)_{b\mathcal{A}, c^{-1}(0)}^{(2)}) = N(L_{a^{-(b, c)}})$. Then,

$$L_{(av)^\#}|_{(av)^{-1}(0)} = (HL_{a^{-(b, c)}})|_{(av)^{-1}(0)} = 0.$$

In addition, according to [10, Remark 1] (see the paragraph that follows Definition 2.6), $L_{a^{-(b, c)}}|_{av\mathcal{A}}^{v\mathcal{A}} = (L_a)_{b\mathcal{A}, c^{-1}(0)}^{(2)}|_{av\mathcal{A}}^{v\mathcal{A}} = (L_a|_{v\mathcal{A}}^{av\mathcal{A}})^{-1}$. Thus,

$$(HL_{a^{-(b, c)}})|_{av\mathcal{A}} = H|_{v\mathcal{A}}^{av\mathcal{A}} L_{a^{-(b, c)}}|_{av\mathcal{A}}^{v\mathcal{A}} = L_{(av)^\#}|_{av\mathcal{A}}.$$

Since $\mathcal{A} = av\mathcal{A} \oplus (av)^{-1}(0)$, $L_{(av)^\#} = HL_{a^{-(b, c)}}$.

Therefore, $(av)^\# = H(a^{-(b, c)})$ and $\|(av)^\#\| \leq \|H\| \|a^{-(b, c)}\|$.

In the following theorem, a bound for $\|a^{-(b, c)} - v(\lambda + av)^{-1}\|$ (Theorem 7.5) will be given. First three particular cases will be considered. Under the same hypotheses of Theorem 7.5, note that if $a^{-(b, c)} = 0$, then according to Definition 2.1, $b = 0 = c$, which in turn implies that $v = 0$. In particular, $v(\lambda + av)^{-1} = 0$. Note in particular that if $a = 0$, then $a^{-(b, c)} = 0$. In addition, if the operator $H \in \mathbb{L}(\mathcal{A})$ defined in Remark 7.7 is such that $H = 0$, then $v = 0$, which implies that $b = 0 = c$ and then, according to Definition 2.1, also $a^{-(b, c)} = 0$. As a result, when $\|a\| \|a^{-(b, c)}\| \|H\| = 0$, $\|a^{-(b, c)} - v(\lambda + av)^{-1}\| = 0$ and no bound is necessary.

Theorem 7.8. Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists. Let $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$. Suppose that

$\| a \| \| a^{-(b, c)} \| \| H \| \neq 0$ and consider $\lambda \in \mathbb{C} \setminus \sigma(-av)$ such that $|\lambda| < \frac{1}{\|a\| \|a^{-(b, c)}\|^2 \|H\|}$, where $H \in L(\mathcal{A})$ is the operator defined in Remark 7.7. Then,

$$\| a^{-(b, c)} - v(\lambda + av)^{-1} \| \leq \frac{|\lambda| \| v \| \| a \| \| a^{-(b, c)} \|^3 \| H \|^2}{1 - |\lambda| \| a \| \| a^{-(b, c)} \|^2 \| H \|}.$$

Proof. According to Lemma 7.1 and [3, Theorem 4.1], $a^{-(b, c)} = a_{bg, 1-hc}^{(2, l)}$, av is group invertible and $a^{-(b, c)} = v(av)^\sharp$. In addition, since $\lambda \in \mathbb{C} \setminus \sigma(-av)$ and $0 \in \sigma(-av)$ ([9, Theorem 4]), $\lambda \neq 0$ and $\lambda + av$ is invertible in \mathcal{A} . Moreover, since $\lambda + av = (\lambda p_{av} + av) + \lambda(1 - p_{av})$ and $av = p_{av}avp_{av}$, $(\lambda + av)^{-1} = (\lambda p_{av} + av)_{p_{av}\mathcal{A}p_{av}}^{-1} + \lambda^{-1}(1 - p_{av})$, where $p_{av} = av(av)^\sharp \in \mathcal{A}^\bullet$ and $(\lambda p_{av} + av)_{p_{av}\mathcal{A}p_{av}}^{-1}$ is the inverse of $\lambda p_{av} + av$ in $p_{av}\mathcal{A}p_{av}$.

Now note that according to Theorem 4.1, if $g \in b\{1\}$ and $h \in c\{1\}$, then

$$av(av)^\sharp = aa^{-(b, c)} = hcabga^{-(b, c)} + (1 - hc)abga^{-(b, c)} = hc + (1 - hc)abga^{-(b, c)}.$$

However, since $v^{-1}(0) = c^{-1}(0) = (hc)^{-1}(0)$, $vav(av)^\sharp = vhc = v$, equivalently $v(1 - p_{av}) = 0$. Consequently,

$$a^{-(b, c)} - v(\lambda + av)^{-1} = v((av)^\sharp - (\lambda + av)^{-1}) = v((av)^\sharp - (\lambda p_{av} + av)_{p_{av}\mathcal{A}p_{av}}^{-1}).$$

In particular,

$$\| a^{-(b, c)} - v(\lambda + av)^{-1} \| \leq \| v \| \| (av)^\sharp - (\lambda p_{av} + av)_{p_{av}\mathcal{A}p_{av}}^{-1} \|.$$

Note that if $(av)^\sharp = 0$, since $a^{-(b, c)} = v(av)^\sharp$, then $a^{-(b, c)} = 0$, which is impossible. In particular, $p_{av} \neq 0$. Now, since $p_{av} = aa^{-(b, c)}$, according to Remark 7.7, $|\lambda| < \frac{1}{\|a\| \|a^{-(b, c)}\|^2 \|H\|} \leq \frac{1}{\|p_{av}\| \| (av)^\sharp \|}$. Furthermore, since $(av)^\sharp$ is the inverse of av in $p_{av}\mathcal{A}p_{av}$, according to Remark 7.6,

$$\| a^{-(b, c)} - v(\lambda + av)^{-1} \| \leq \| v \| \frac{|\lambda| \| p_{av} \| \| (av)^\sharp \|^2}{1 - |\lambda| \| p_{av} \| \| (av)^\sharp \|}.$$

Therefore, since $|\lambda| < \frac{1}{\|a\| \|a^{-(b, c)}\|^2 \|H\|}$, according to Remark 7.7,

$$\| a^{-(b, c)} - v(\lambda + av)^{-1} \| \leq \frac{|\lambda| \| v \| \| a \| \| a^{-(b, c)} \|^3 \| H \|^2}{1 - |\lambda| \| a \| \| a^{-(b, c)} \|^2 \| H \|}.$$

□

To establish a bound for $\| a^{-(b, c)} - (\lambda + va)^{-1}v \|$ (Theorem 7.5), it is necessary to note some facts first.

Remark 7.9. Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b, c)}$ exists. Let $v \in \mathcal{A}$ be such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$. Then, as in the proof of Theorem 7.3, the (c, b) -inverse of $a \in \mathcal{A}_{op}$ exists and coincide with $a^{-(b, c)}$. Moreover, $v \in \mathcal{A}_{op}$ is such that $v \cdot_{op} \mathcal{A} = c \cdot_{op} \mathcal{A}$ and $v^{-1}(0) = b^{-1}(0)$ (in \mathcal{A}_{op}). Thus, since $a^{-(b, c)} - (\lambda + va)^{-1}v = a^{-(b, c)} - v \cdot_{op} (\lambda + a \cdot_{op} v)^{-1}$, to give a bound for $\| a^{-(b, c)} - (\lambda + va)^{-1}v \|$, it is enough to apply Theorem 7.8 with the operator $\tilde{H} \in L(\mathcal{A})$, which is the map defined in Remark 7.7 that corresponds to the (c, b) -inverse of a in \mathcal{A}_{op} . Thus, $\tilde{H} \in L(\mathcal{A})$ is defined as follows:

$$\tilde{H}|_{(1-hc)\mathcal{A}} = 0, \quad \tilde{H}|_{\mathcal{A}v}^{\mathcal{A}va} = (R_v|_{\mathcal{A}va}^{\mathcal{A}v})^{-1}: \mathcal{A}v \rightarrow \mathcal{A}va.$$

Note that when $\|a\| \|a^{-(b,c)}\| \|\tilde{H}\| = 0$, $a^{-(b,c)} = 0$, $v = 0$ and $\|a^{-(b,c)} - (\lambda + va)^{-1}v\| = 0$, so that no bound is needed.

Corollary 7.10. *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$ and $b, c \in \hat{\mathcal{A}}$ such that $a^{-(b,c)}$ exists. Let $v \in \mathcal{A}$ such that $v\mathcal{A} = b\mathcal{A}$ and $v^{-1}(0) = c^{-1}(0)$. Suppose that $\|a\| \|a^{-(b,c)}\| \|\tilde{H}\| \neq 0$ and consider $\lambda \in \mathbb{C} \setminus \sigma(-va)$ such that $|\lambda| < \frac{1}{\|a\| \|a^{-(b,c)}\|^2 \|\tilde{H}\|}$, where $\tilde{H} \in L(\mathcal{A})$ is the operator defined in Remark 7.9. Then,*

$$\|a^{-(b,c)} - (\lambda + va)^{-1}v\| \leq \frac{|\lambda| \|v\| \|a\| \|a^{-(b,c)}\|^3 \|\tilde{H}\|^2}{1 - |\lambda| \|a\| \|a^{-(b,c)}\|^2 \|\tilde{H}\|}.$$

Proof. Apply Theorem 7.8 and Remark 7.9. \square

Next the continuity of the (b,c) -inverse inverse will be studied. In first place, a technical lemma will be presented.

Lemma 7.11. *Let \mathcal{A} be a Banach algebra and consider $a, b \in \mathcal{A}$ and $p, q, r, s \in \mathcal{A}^\bullet$ such that $a^{-(p,q)}$ and $b^{-(r,s)}$ exist. Then*

$$\begin{aligned} b^{-(r,s)} - a^{-(p,q)} &= b^{-(r,s)}(s - q)(1 - aa^{-(p,q)}) - (1 - b^{-(r,s)}b)(r - p)a^{-(p,q)} \\ &\quad + b^{-(r,s)}(a - b)a^{-(p,q)}. \end{aligned}$$

Proof. Recall that according to Theorem 4.1, $a^{-(p,q)} \in p\mathcal{A}q$ and $b^{-(r,s)}br = r$. Thus,

$$\begin{aligned} b^{-(r,s)}ba^{-(p,q)} - a^{-(p,q)} &= -(1 - b^{-(r,s)}b)pa^{-(p,q)} \\ &= [(1 - b^{-(r,s)}b)r - (1 - b^{-(r,s)}b)p]a^{-(p,q)} \\ &= (1 - b^{-(r,s)}b)(r - p)a^{-(p,q)}. \end{aligned}$$

In addition, according again to Theorem 4.1, $b^{-(r,s)} \in r\mathcal{A}s$ and $qaa^{-(p,q)} = q$. In particular,

$$\begin{aligned} b^{-(r,s)} - b^{-(r,s)}aa^{-(p,q)} &= b^{-(r,s)}s(1 - aa^{-(p,q)}) \\ &= b^{-(r,s)}[s(1 - aa^{-(p,q)}) - q(1 - aa^{-(p,q)})] \\ &= b^{-(r,s)}(s - q)(1 - aa^{-(p,q)}) \end{aligned}$$

Therefore,

$$\begin{aligned} b^{-(r,s)} - a^{-(p,q)} &= b^{-(r,s)}(s - q)(1 - aa^{-(p,q)}) + b^{-(r,s)}aa^{-(p,q)} \\ &\quad - (1 - b^{-(r,s)}b)(r - p)a^{-(p,q)} - b^{-(r,s)}ba^{-(p,q)} \\ &= b^{-(r,s)}(s - q)(1 - aa^{-(p,q)}) - (1 - b^{-(r,s)}b)(r - p)a^{-(p,q)} \\ &\quad + b^{-(r,s)}(a - b)a^{-(p,q)}. \end{aligned}$$

\square

In the following Corollary the (b, c) -inverse case will be considered.

Corollary 7.12. *Let \mathcal{A} be a Banach algebra and consider $a_i \in \mathcal{A}$ and $b_i, c_i \in \mathcal{A}^\bullet$ such that $a_i^{- (b_i, c_i)}$ exist ($i = 1, 2$). Consider $g_i \in b_i\{1\}$ and $h_i \in c_i\{1\}$ ($i = 1, 2$). Then*

$$\begin{aligned} a_2^{- (b_2, c_2)} - a_1^{- (b_1, c_1)} &= a_2^{- (b_2, c_2)} (h_2 c_2 - h_1 c_1) (1 - a_1 a_1^{- (b_1, c_1)}) \\ &\quad - (1 - a_2^{- (b_2, c_2)} a_2) (b_2 g_2 - b_1 g_1) a_1^{- (b_1, c_1)} + a_2^{- (b_2, c_2)} (a_1 - a_2) a_1^{- (b_1, c_1)}. \end{aligned}$$

Proof. According to Proposition 5.1,

$$a^{- (b_1, c_1)} = a^{- (b_1 g_1, h_1 c_1)}, \quad a_2^{- (b_2, c_2)} = a_2^{- (b_2 g_2, h_2 c_2)}.$$

Now apply Lemma 7.11 for $a = a_1$, $b = a_2$, $p = b_1 g_1$, $q = h_1 c_1$, $r = b_2 g_2$ and $s = h_2 c_2$. \square

Now the continuity of the (b, c) -inverse will be characterized.

Theorem 7.13. *Let \mathcal{A} be a Banach algebra and consider $a \in \mathcal{A}$, $b, c \in \hat{\mathcal{A}}$, $g \in b\{1\}$ and $h \in c\{1\}$ such that $a^{- (b, c)}$ exists. Suppose that there are sequences $(a_n)_{n \geq 1} \subset \mathcal{A}$, $(b_n)_{n \geq 1} \subset \hat{\mathcal{A}}$, $(g_n)_{n \geq 1} \subset \mathcal{A}$, $(c_n)_{n \geq 1} \subset \hat{\mathcal{A}}$ and $(h_n)_{n \geq 1} \subset \mathcal{A}$ such that for each $n \in \mathbb{N}$, $g_n \in b_n\{1\}$, $h_n \in c_n\{1\}$, $a_n^{- (b_n, c_n)}$ exists and $(a_n)_{n \geq 1}$ (respectively $(b_n g_n)_{n \geq 1}$, $(h_n c_n)_{n \geq 1}$) converges to a (respectively to bg , hc). Then, the following statements are equivalent.*

- (i) *There exists $k \in \mathbb{R}$ such that $\| a_n^{- (b_n, c_n)} \| \leq k$.*
- (ii) *The sequence $a_n^{- (b_n, c_n)}$ converges to $a^{- (b, c)}$.*

Proof. It is enough to prove that statement (i) implies statement (ii). To this end, apply Corollary 7.12. \square

REFERENCES

1. J. Benítez and E. Boasso, The inverse along an element in rings, arXiv: 1507.05410.
2. F. F. Bonsall and J. Duncan, Complete Normed Algebras, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
3. J. Cao and Y. Xue, The characterizations and representations for the generalized inverses with prescribed idempotents in Banach algebras, Filomat 27 (2013), 851-863.
4. N. Castro-González, J. Chen and L. Wang, Further results on generalized inverses in rings with involution, Electron. J. Linear Algebra 30 (2015), 118-134.
5. M. P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012), 1909-1923.
6. M. P. Drazin, Commuting properties of generalized inverses, Linear Multilinear Algebra, 61 (2013), 1675-1681.
7. G. Kantún-Montiel, Outer generalized inverses with prescribed ideals, Linear Multilinear Algebra 62 (2014), 1187-1196.
8. Y. Ke and J. Chen, The Bott-Duffin (e, f) -inverses and their applications, Linear Algebra Appl. 489 (2016), 61-74.
9. C. King, A note on Drazin inverses, Pacific J. Math. 70 (1977), 383-390.
10. X. Liu, Y. Yu, J. Zhong and Y. Wei, Integral and limit representations of the outer inverse in Banach space, Linear Multilinear Algebra 60 (2012), 333-347.
11. D. Mosić, D. S. Djordjević and G. Kantún-Montiel, Image-kernel (P, Q) -inverses in rings, Electron. J. Linear Algebra 27 (2014), 272-283.

12. Y. Yu and Y. Wei, The representation and computational procedures for the generalized inverse $A_{T, S}^{(2)}$ of an operator A in Hilbert spaces, Numer. Funct. Anal. Optim. 30 (2009), 168-182.

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